91. Surface Tension of Drops and Jets

I. Introduction

In the first part of this experiment, the surface tension of water and other liquids is determined by using the pendant drop technique. In the second part, the shape of a water jet from a tap is analysed. Using energy conservation and the continuity equation, an analytic expression for the shape of the water jet can be found, and compared to experimental findings.

1. Requirements

- Smartphone or camera
- Computer for the data analysis
- Pipette (if possible), otherwise a thin straw to have drops pending from it
- 3-4 different, common liquids other than water, e.g. oil, sirup, ethanol, soapy water, ...[1]
- A water tap
- Ruler or another object of known dimension used as scale

II. Basics

If a drop of a liquid is suspended from a pointy object, it resides in a characteristic shape that resembles a pear. The dimensions and the exact shape greatly depend on the surface tension of the fluid. The surface tension is the force which prevents a drop from “falling apart” and forces the liquid to form a drop.
On a molecular level, the surface tension can be understood by looking at molecules inside the liquid and at the border of the liquid (see also Fig. 1). A molecule inside the fluid is surrounded by other molecules of the fluid, so that the forces (electrical attraction forces between atoms and molecules) cancel each other out in a state of equilibrium. This force holding the fluid together is called the cohesion force. For a molecule at the interface,the cohesive forces no longer cancel each other out, but there is an inwardly directed resulting force $F_R \neq 0$, as shown in Fig. 1. Thus, a force must be applied to bring a molecule to the surface of the fluid. Consequently, a suspended drop of liquid tries to form a perfect sphere (which has minimal surface compared to the volume), but gets disturbed by the gravitational force pulling each molecule downwards.

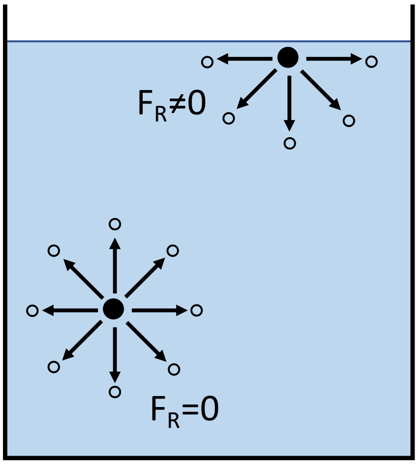

Fig. 1: No resulting molecular force acts on a molecule inside the fluid. But on a molecule at the edge of the fluid, there is a net resulting molecular force, pulling the molecule back into the fluid.

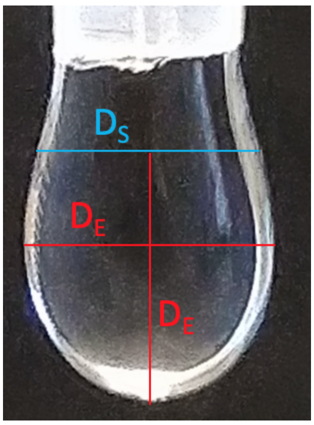

Fig. 2: Photo taken with a smart phone of a drop of pure water. The relevant dimensions D_E and D_S are marked already (see later in the text).

[1]If you have high-concentration ethanol (ideally close to 100%, but everything > 40% should work as well), you can also measure the surface tension of an ethanol-water mixture for different concentrations. Start with 100% water, then 10% ethanol and 90% water, followed by 20% ethanol and 80% water etc.

1. Forces acting on a suspended drop

In a drop of liquid, there is a pressure difference between the liquid and the outside. This is even true for more general interfaces between different substances, however, here the focus is put on a drop of liquid suspended in air. Additional information and further reading is provided in [1, 3, 6].
The pressure difference between at the liquid/gas-interface depends on the surface tension and the curvature of the drop. The **Laplace-Young-equation** describing this relation reads

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right), \tag{1}$$

where ΔP is the pressure difference between the two interfaces (sometimes called **Laplace pressure**), γ is the surface tension and R_1 and R_2 are the two principal radii of curvature, respectively. The derivation of the equation in the case of a sphere ($R_1 = R_2$) can be found in the appendix.
The forces acting on a suspended drop are the gravitational force, trying to pull down the molecules inside the drop, and the surface energy forces which try to keep the liquid together, resulting in the surface tension. The gravitational force leads to an inhomogeneous pressure distribution inside the drop: the hydrostatic pressure, which depends on the position z along the vertical axis:

$$\Delta P(z) = P_0 \pm \Delta\rho g z, \tag{2}$$

where P_0 is a constant, $\Delta\rho$ the density difference between the density of the liquid and the surrounding environment, and g is the gravitational acceleration. Equating the two formulae 1 and 2 we obtain:

$$\gamma\,(1/R_1 + 1/R_2) = P_0 \pm \Delta\rho g z \tag{3}$$

Let now x be the horizontal axis, and z as before the coordinate of any point on the symmetry axis (see Fig. 3). We define R' as the radius of curvature of the meridional section at height z, and φ as the angle which the normal to the drop surface forms with the symmetry axis.

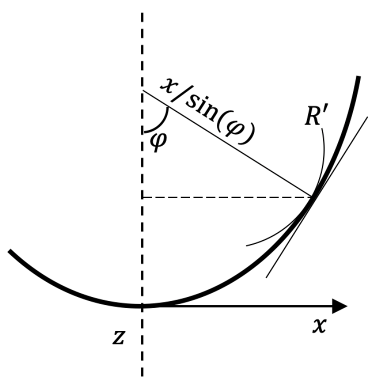

Fig. 3: Coordinate system. R' is the radius of curvature of the meridional section at height z.

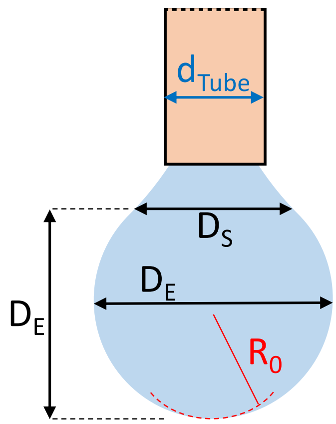

Fig. 4: Sketch of a drop with the necessary dimensions. D_E and D_S are the diameters at the equator and a distance D_E from the apex, respectively. And R_0 is the radius at the apex.

From Fig. 3 we see that the two principal radii are $R_1 = R'$ and $R_2 = x/\sin(\varphi)$. Thus, the above formula 3 becomes

$$\frac{1}{R'} + \frac{\sin(\varphi)}{x} = \frac{P_0 + \Delta\rho g z}{\gamma} \tag{4}$$

At the apex ($z \to 0$), the two principal radii R_1 and R_2 are the same, thus $R_1 = R_2 = R_0$, which makes the apex a natural choice of the coordinate's origin. This means that $P_0/\gamma = 2/R_0$, and hence

$$\frac{1}{R'} + \frac{\sin(\varphi)}{x} = \frac{2}{R_0} + \frac{\Delta\rho g z}{\gamma} \tag{5}$$

Multiplying with R_0 we can write

$$\frac{R_0}{R'} + \frac{\sin(\varphi)}{x/R_0} = 2 + \frac{\Delta\rho g R_0^2}{\gamma}\frac{z}{R_0} = 2 + \beta z \tag{6}$$

We then can take the quantity R_0 as the unit of length, so formally $\bar{x} = x/R_0$, $\bar{z} = z/R_0$ and $\bar{R} = R/R_0$. For the sake of simplicity we drop the bar ($\bar{x} \to x$ etc.), and we obtain

$$\frac{1}{R'} + \frac{\sin(\varphi)}{x} = 2 + \frac{\Delta\rho g R_0^2}{\gamma} z. \tag{7}$$

It can be seen now that the shape of an axisymmetric pendant drop depends on solely one quantity β:

$$\beta \equiv \Delta\rho g R_0^2/\gamma \tag{8}$$

The quantity β is a measure of the relative magnitudes between the gravitational force and the interfacial force, and is sometimes called "**Bond** number".
If it was possible to accurately determine the Bond number from the shape of a drop of liquid, together with the radius of curvature at the apex, then the interfacial tension would be easy to extract by the above equation. Unfortunately, determining the Bond number for a given system is difficult, and was even more difficult in the past. But J. M. Andreas devised a simpler approach [5] for determining this quantity.
Instead of taking the radius of curvature, it is also possible to measure the maximum drop diameter D_E, and the diameter D_S at a position D_E above the apex, see also Fig. 4. This ratio shall be defined as $S \equiv D_S/D_E$. The ratio S can then be compared to tabulated values [6], which relate the radius of curvature at the apex, R_0, with the two quantities D_S and D_E. The relation between the Bond number β and S can in general be written as:

$$\frac{1}{\beta} = f\left(\frac{D_S}{D_E}\right) \tag{9}$$

By fitting equation 9 to tabulated values, we note that the function $1/\beta$ can be approximated to a good extent by the simple analytical formula

$$\frac{1}{\beta} = a\left(\frac{D_S}{D_E}\right)^b \tag{10}$$

with $a \approx 0.345$ and $b \approx -2.5$. Therefore, the surface tension can be determined using equation 8 and 10:

$$\boxed{\gamma = a\Delta\rho g D_E^2 \left(\frac{D_S}{D_E}\right)^b} \tag{11}$$

2. Water from a tap

A water jet flowing out of a tap shows a characteristic shape as shown in Fig. 5. The shape can be described by a formula which can be easily derived from two fundamental physics laws: the energy conservation and the continuity equation.
The former states that, in absence of friction, the total energy U of an isolated system is conserved, and it is the sum of the potential energy U_{pot} and the kinetic energy U_{kin}, i.e.:

$$U(z) = U_{\text{pot}}(z) + U_{\text{kin}}(z) = M \cdot g \cdot (-z) + \frac{1}{2} \cdot M \cdot v^2(z) \tag{12}$$

where M is the mass, v is the speed and g is the gravitational acceleration. The negative sign of the z coordinate in $U_{\text{pot}}(z)$ derives from the chosen system of reference, as described in Fig.5. In order to describe the water flow, we need to follow a hypothetically, **infinitesimally thin** slice along its path from the tap down to the ground. At the starting position $z = z_0$, the slice has a mass m_{sl} and a speed $v(z_0) = v_0$. Following Eq. (12), its total energy at position z_0 is

$$U(z = z_0) = U_0 = -m_{\text{sl}} \cdot g \cdot z_0 + \frac{1}{2} \cdot m_{\text{sl}} \cdot v_0^2 \tag{13}$$

When the same slice, with the same mass m_{sl}, reaches a position $z > z_0$, its total energy is provided by:

$$U(z) = -m_{\text{sl}} \cdot g \cdot z + \frac{1}{2} \cdot m_{\text{sl}} \cdot v^2(z) \tag{14}$$

In this water jet model, friction is neglected, and energy is conserved. In this way, the speed of the water slice as function of its distance z from the tap can be derived:

$$U(z) = U_0$$
$$-m_{\text{sl}} \cdot g \cdot z + \frac{1}{2} \cdot m_{\text{sl}} \cdot v^2(z) = -m_{\text{sl}} \cdot g \cdot z_0 + \frac{1}{2} \cdot m_{\text{sl}} \cdot v_0^2$$

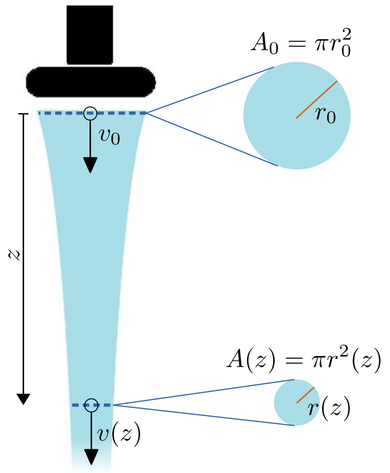

Fig. 5: Characteristic shape of a water jet flowing from a tap. Important physical quantities are highlighted. Note that the z-axis is oriented downwards.

And solving for $v(z)$:

$$v(z) = \sqrt{v_0^2 + 2 \cdot g \cdot z} \tag{15}$$

This equation still does not provide any information about the shape of the water jet, and in particular about its horizontal extension as function of the z position. To derive it, the continuity equation for fluids is exploited. In particular, it can be demonstrated that, for an incompressible fluid, the product $A(z) \cdot v(z)$ is conserved, i.e.:

$$A_0 \cdot v_0 = A(z) \cdot v(z) \tag{16}$$

where v is the speed of the fluid, and A is the area of the section orthogonal to the fluid propagation direction, as highlighted in Fig. 5. Eq. (16) can be expanded and inverted, to get $r(z)$, that is the radius of the jet as function of the distance from the tap:

$$\begin{aligned} \pi \cdot r_0^2 \cdot v_0 &= \pi \cdot r(z)^2 \cdot v(z) \\ r(z) &= r_0 \sqrt{\frac{v_0}{v(z)}} \end{aligned} \tag{17}$$

Finally, Eq. (15) and (17) can be combined to get the shape of the water jet:

$$\boxed{r(z) = \frac{r_0}{\sqrt[4]{1 + \frac{2 \cdot g \cdot z}{v_0^2}}}} \tag{18}$$

It is also possible to invert Eq. (18) to get the starting velocity of the water flow:

$$\boxed{v_0^2 = 2g \cdot z \cdot \left[\frac{1}{1 - \Gamma^4} - 1\right] \quad \text{with} \ \ \Gamma = \frac{r(z)}{r_0}} \tag{19}$$

III. Tasks

1. Determine the surface tension for 4 different liquids, one of them should be water.
2. Compare the results to literature values and discuss the sources of uncertainties.
3. On what physical / chemical properties does the surface tension depend? Does your data support this explanation?
4. Use another “tip” to suspend the drops of water, and determine the surface tension measured in that way. Are the results equally accurate, and if not, why?
5. Take a photo of water flowing out of your tap at home. Using Equation (19), calculate the initial velocity v_0 of the water from your tap. To do so, make different estimations of v_0, by measuring the radius of the jet at different z positions. Are the v_0 values compatible with each other? Which is the uncertainty of their average value $< v_0 >$?

6. **For students of D-PHYS**: Insert the computed average v_0 in Eq. (18), and plot the function $r(z)$ along with the radii experimentally measured in the previous task. Fit the data to the model. Does $r(z)$ fit well the experimental data points?

7. **For the interested and ambitious students of D-PHYS**: Read the pdf of MIT about the The Plateau-Rayleigh Instability `http://web.mit.edu/1.63/www/Lec-notes/Surfacetension/Lecture5.pdf`. Can you observe these instabilities as well? Can you estimate the breakup-time (defined in Eq. 31 in the linked pdf), and does it correspond to your observations?

IV. Conducting the experiments

About the different liquids

Some suggested liquids are: Water, ethanol or alcohol (as pure as possible), oil, highly concentrated sugar or salt water, soap water, some gels (e.g. glycerine, shower gel), honey. You are of course free to think about more liquids. Even "complicated" liquids like hot wax are very interesting, since their surface tension depends on the temperature.

1. Water drop

The measurements should be performed by taking a picture of the drop with a smart phone or a digital camera. The capillary, from which the drop is suspended, has to be fixed in order to be able to take good quality pictures. Then, enlarge the picture and measure the two distances D_E and D_S as accurately as possible. Make an estimation of the precision of your measurement. Repeat the measurement 10 times before changing the liquid. A good choice of liquids might be: Water, oil, water with detergent or soap, ethanol, sugary solutions or similar. Make sure you can find the densities of the liquids you used, or measure them (which is possible with commercial means, no hydrometer (=aräometer in German) is required!). Assume reasonable values for the ambient air density (depends on the temperature!).
In order to get nicely defined drop shapes, it is imperative to use a capillary which is as thin as possible (i.e., D_{tube} should best be < 2 mm). Ideally, a pasteur pipette or a syringe is used. Fix the pipette or syringe, and make sure the illumination is adequate.

2. Water flow

Again, the experiment is conducted by taking pictures of the water flowing from the tap. For this part, a reference dimension is needed. A ruler should be placed close to the water jet. Then, the dimensions should be measured by "counting" pixels, with the ruler serving as reference. It is far less precise to use the ruler to measure directly the width of the water jet. Alternatively, an object of known length in the picture can be used to measure the diameter of the water jet. Also the outlet diameter of the tap could provide this information.
It is important that the tap is sufficiently wide open, such that a nice water jet is formed. If the jet consists merely of a "dripple", the formulas are not valid because the initial radius does not correspond to the tap outlet. Note that many taps have a filter at the outlet. Does this influence your measurement, and if yes, how?

V. Appendix

Derivation of equation 1

The surface A of a sphere or drop of liquid with diameter r is

$$A = 4\pi r^2, \tag{20}$$

and its volume is

$$V = 4/3\pi r^3. \tag{21}$$

If the radius is altered by a small amount $\mathrm{d}r$, the change of the surface and volume, respectively, is:

$$\begin{aligned} \mathrm{d}A &= 8\pi r \mathrm{d}r \\ \mathrm{d}V &= 4\pi r^2 \mathrm{d}r. \end{aligned} \tag{22}$$

The work $\mathrm{d}W$ it takes to change the surface is

$$\mathrm{d}W = \gamma \mathrm{d}A = \gamma 8\pi r \mathrm{d}r, \tag{23}$$

where γ is the surface tension, which acts against the change of the surface. For changing the volume, work has to be invested against the pressure p, thus analogously we get:

$$\mathrm{d}W = p\mathrm{d}V = p4\pi r^2\mathrm{d}r. \tag{24}$$

Equating the two expresions yields the Laplace-Young equation for a sphere, which is equivalent to equation 1 with $R_1 = R_2 = r$.

Literatur

[1] P. S. Laplace *Mecanique celeste*, supplement to the 10th book Duprat, Paris, 1806.

[2] C. E. Stauffer *The measurement of surface tension by the pendant drop technique*, J. Phys. Chem. **69**, 1933 (1965).

[3] F, Bashforth and J. C. Adams, *An attempt to test the theories of capillary action* University Press, Cambridge, England, 1883.

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