90. Helmholtz Resonator

The goal of this experiment is to measure the speed of sound by using a simple universal Helmholtz Resonator, made from a bottle with different water levels. By blowing over the bottle's neck and exciting the resonance frequency, it is possible to extract the resonance frequency as a function of filling level. Using the Helmholtz Resonance Equation, the speed of sound can be extracted.

I. Required equipment

- An empty bottle (glass, plastic) which should be as close as possible to two cylinders (see Fig. 2)
- A smartphone/tablet with the possibility to record acoustic spectra
- A scale or water meter
- A ruler to measure the dimensions of the bottle

II. Introduction

A Helmholtz Resonator is an acoustic device which was developed in the 1850s by **Hermann von Helmholtz** to single out an individual frequency (the resonant frequency) from a broad spectrum of frequencies, typically music or other complex sounds.
The phenomenon is known to everybody: If you blow over the top of an empty bottle, you can excite a single frequency, which is the resonant frequency for that bottle. While the original Helmholtz Resonator had fixed dimension and thus a fixed resonance frequency, the concept of it can be further developed into the so-called **u**niversal Helmholtz Resonator: If the bottle is not fully empty, the resonant frequency (or the pitch level) is different. It is therefore possible to excite the universal Helmholtz Resonator at different frequencies by changing the filling level of a bottle. The filling level defines the cavity volume of the resonator, because the water is incompressible and represents a hard boundary.

III. Derivation of the Helmholtz Equation

1. Preliminary considerations

In order to derive the Helmholtz equation, we start with a modified thought experiment. A small mass m is inserted into a bottle's neck (see also Fig. 1), which has a cross-sectional area A. The mass is thought to seal the bottle, so that no air enters or leaves the bottle's neck. Thus, the mass will drop until it compresses the air enough such that the counteracting pressure P exerts a force F onto the mass, supporting its weight. The compressed air in the bottle's neck can be though of as a spring with spring constant k, thus the system is a simple harmonic oscillator. If friction is neglected, the mass will oscillate around its equilibrium position y_0 with angular frequency ω. The spring force F is proportional to the displacement from the equilibrium position, $F = -k(y - y_0)$, so that we can write:

$$\begin{aligned}\omega^2 &= \frac{k}{m} \\ &= -\frac{1}{m}\frac{dF}{dy}\bigg|_{y_0} \\ &= -\frac{A}{m}\frac{dP}{dy}\bigg|_{y_0} \\ &= -\frac{A^2}{m}\frac{dP}{dV}\bigg|_{V_0}\end{aligned}$$

In the third line, we replaced $F = AP$ and the volume V of the bottle neck by $V = Ay$. The index "0" indicates the equilibrium (or static) volume and pressure, respectively. Neglecting thermal effects of compressed gases, we know from thermodynamics that (for adiabatic processes) it holds: $PV^\gamma = const$, where γ is the isentropic expansion factor or adiabatic index (usually 1.4 for air). This immediately leads to the expression $dP/dV = -\gamma P/V$. Thus:

$$\omega^2 = \gamma \frac{A^2}{m}\frac{P_0}{V_0}. \tag{1}$$

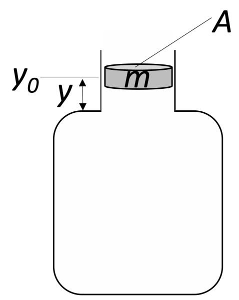

Fig. 1: Sketch of a mass m with cross-section area A oscillating around the equilibrium position y_0.

2. Expanding the discussion to a real bottle

For a real bottle with a cylindrical bottle neck, we have $A = V_n/L'$, where L' is the equivalent length of the neck with an end-correction. The end-correction is not trivial to derive, and is given here as $L' = L + cD/2$ with $c = 1.46212$ (see [1] and [2] for more details). Simply put it accounts for the fact that standing sound waves can penetrate the ambient volume a bit.
We therefore find from eq. 1

$$\omega^2 = \gamma \frac{A}{m} \frac{V_n}{L'} \frac{P_0}{V_0}. \tag{2}$$

where V_0 refers to the air volume of the resonator from the water level to the beginning of the neck. By definition, the mass density ρ is equal to m/V, and the speed of sound in a gas is given by $v = \sqrt{\gamma P_0/\rho}$. Putting all together, we arrive at

$$\omega^2 = v^2 A \frac{1}{V_0 L'}. \tag{3}$$

Converting the angular frequency ω to the frequency $f = \omega/2\pi$, we obtain the resonance frequency of our Helmholtz Resonator:

$$f = \frac{v}{2\pi} \sqrt{\frac{A}{V_0 L'}}. \tag{4}$$

All the above expressions are valid as long as the wavelength λ is much larger than any relevant dimensions: $\lambda \gg L, D, h$ with h being the height of the resonant cavity, which is true in this case.

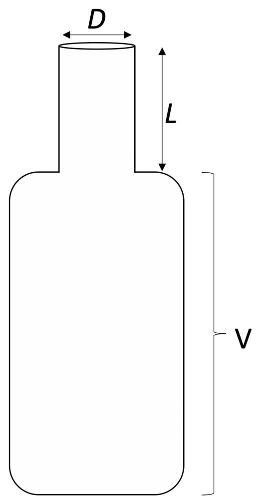

Fig. 2: Sketch of an idealized bottle with neck diameter D, with cross-section area of the neck $A = \pi(D/2)^2$ and length L. In case of an empty bottle, the resonator volume is V.

IV. Tasks

Measure the resonance frequency of the Helmholtz Resonator as a function of the cavity volume (i.e., the filling level) of a bottle, for at least 10 cavity volumes. Make sure to not fill the bottle by more than 50 percent. From these measured values, extract the speed of sound in air and compare it to the literature value. In order to do so, plot f^2 as a function of $1/V$ and fit a straight line to it. Show the plot and the fit in your report.

V. Conduction of the experiment

For this experiment, a bottle is needed which can be approximated as two cylinders, as shown in Fig. 2 or Fig. 3. Glass bottles are in general preferable compared to plastic bottles, but both work fine. Note that it is not trivial for most bottles to determine V and L precisely. This will also add substantially to the uncertainty of the measurement. Moreover, a tablet or smartphone with a dedicated app is needed, such that it is possible to record an acoustic spectrum. A good app which is handy for this purpose is the “PhyPhox App” or the “Physics Toolbox Suite”, both can be downloaded for free.
In a first step, the dimensions of the bottle are measured. The volume V of the resonator is also determined for the empty bottle. This can be done by putting the bottle on a scale and setting it to zero. Then, the resonator volume is filled with water until the neck. Read the scale, and determine the volume which is now filled with water. When the bottle is filled with more and more water, the relevant cavity volume V will decrease. For small cavity volumes (i.e., when the bottle is almost filled with water) the determination of the cavity volume becomes less accurate. Therefore, the measurement for large cavity volumes (low water levels) are more reliable.
Next, the bottle is filled with different amounts of water, yielding different cavity volumes. For each amount of water, blow over the bottle neck and record the audio spectrum. There should be a clearly visible maximum amplitude at the corresponding resonance frequency. In the example shown in Fig. 3, the "choose positionoption of the app was used to read out the position of the maximum amplitude (in the example in Fig. 3 it is at 609 Hz). Make sure that your app has a sufficient fine resolution. In the PhyPhox-App, you can go to “settings” and then change the resolution. This is important since the default is about 2000 data points, distributed over a wide range up to several kHz frequency. Therefore, the resolution in the relevant range is too poor (>5 Hz).
Write down for every filling level the resonance frequency. If you have at least ten different readings, plot the square of the resonance frequency, f^2, as a function of $1/V$. According to equation 4, this should yield a straight line. Fitting a linear function to it and extracting the slope s allows for the calculation of the speed of sound in air v:

$$\text{slope } s = \frac{v^2 A}{4\pi^2 L'} = \frac{v^2 D^2}{16\pi(L + cD/2)} \tag{5}$$

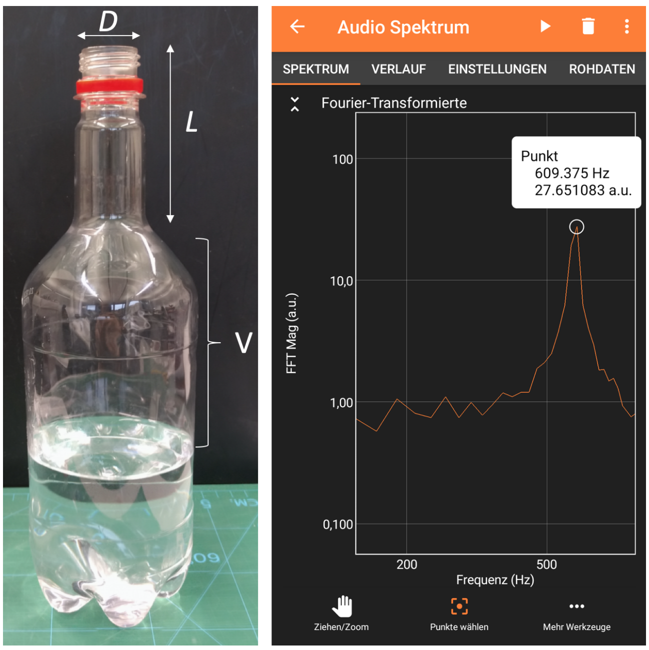

Fig. 3: Half-full bottle with the relevant dimensions (left) and a screenshot from the PhyPhox-App with a recorded spectrum. Note that it is not always trivial to determine the actual length of the neck L.

Literatur

[1] H. Levine and J. Schwinger, *On the radiation of sound from an unflanged circular pipe*, Phys. Review 73 (4) (1948)

[2] L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, 4th edition, Wiley-VCH, (1999)